component equipment conditions for more realistic design and control decisions.

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Finite-Difference Solution of Two Variable Thermal and Mechanical Deformation Problems

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A numerical solution of thermoelastic stress and deformation problems governed by two linear, second-order, elliptic partial differential equations in two unknowns is presented. The analysis is applicable to both axially symmetric solids and variable thickness plates with temperature-dependent material properties. Body forces caused by rotation and/or temperature gradients and boundary conditions corresponding to any combination of prescribed tractions and displacements are considered. A general digital computer program based upon a finite-difference discretization and utilizing the method of successive overrelaxation is briefly described. Selected results from turbomachinery problems and a numerical comparison with the classical rotating ellipsoid are included.

Nomenclature

h	= network spacing
i, j	= network point
l, m	= direction numbers of outward normal with respect to the R and Z axes
r, z	= point coordinates, in.
t	= plate thickness, in.
u, w	= displacements in R and Z directions, respectively, in.
x, y	= coordinates relative to boundary point in R and Z directions, respectively, in.
A_{pq}	= general matrix element
C_1, C_2, C_3	= dimensionless material constants
E	= modulus of elasticity, psi
F_1, F_2	= prescribed boundary values
F_r, F_z	= body forces per unit volume in R and Z directions, respectively, lb/in.^{3}
G	= shear modulus, psi
T	= temperature rise, °F
α	= coefficient of thermal expansion, in./in./°F
α_{kl}, β_{lk}	= matrix elements
δ_p, δ_p°	= elements of displacement and load vectors, respectively, in.
λ	= Lamé constant, psi = $\nu E/(1 + \nu)(1 - 2\nu)$
ν	= Poisson's ratio
ρ	= mass density, lb-sec^2/in.^{4}
σ_r, σ_z	= normal stresses in R and Z directions, respectively, psi
σ_t	= circumferential stress in axisymmetric solid, psi
τ	= shearing stress in R-Z plane, psi
ω	= rotational speed, rad/sec, overrelaxation factor

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1. Introduction

A NUMBER of stress and deformation problems that occur in engineering practice can be represented mathematically by linear, second-order, elliptic partial differential equations. In particular, the extensional problem of the thin, variable thickness plate and the problem of the axisymmetric, three-dimensional solid can be so represented. Thus, the mathematical models of these distinctly different physical bodies, formulated in different geometries, can be incorporated within a general set of elliptic equations that includes these models as special cases.

There is a great deal to be gained by this single formulation, for the mathematical difficulties inherent in a consideration of irregular configurations, mixed boundary conditions, and the effect of thermal loading on material properties arise in both the plate and the axisymmetric solid. In such stress and deformation problems as occur in turbomachinery elements (e.g., turbine disks and blades, axisymmetric nozzles, etc.), thick-shell pressure vessels, and encased solid propellants, for example, these effects of configuration, temperature, and constraint can be quite important. Thus, some sort of discretization is mandatory if the elliptic equations to be solved are to be representative of physically significant problems. To this purpose the finite-difference method is chosen herein as the most feasible means of solution, affording considerable flexibility and enabling the development of a computer program applicable to a wide range of problems.

In recent years, numerical analyses of stress problems in irregular, thick-walled pressure vessels and encased solid propellants have been given by Thoms,[1] Conte, Miller, and Sensenig,[2] Bijlaard and Dohrmann,[3,4] and Hubka.[5] In the field of mechanical and thermal problems in rotating machinery (including blade problems) analyses have been given by Hodge and Papa,[6] Chang,[7] Schilhansl,[8] Kobayashi and Trumpler,[9] and Fried, Reichenbach, and Chingari.[10–12] The most general elasticity formulations are those by Thoms[1] and Kobayashi and Trumpler,[9] using Southwell stress functions, and Conte, Miller, and Sensenig,[2] Hubka,[5] and Fried,[12] using deformations as unknowns. The other papers make use of various approximations appropriate to the particular problems under consideration. The effect of temperature on material properties is included only in the analysis given by Chang.[7] A variational formulation of the difference equations for axisymmetric solids is contained in the paper by Conte, Miller, and Sensenig.

2. General Differential Equations

Beginning with the equations of linearized, uncoupled thermoelasticity that define the thermal-mechanical stress and deformation problem in the interior of an axisymmetric solid[13] (Fig. 1), and considering the variation of material properties E and α with temperature, the final differential equations that govern the displacements u and w can be written

$$(\lambda + 2G)\frac{\partial^2 u}{\partial r^2} + (\lambda + 2G)\left(\frac{1}{E}\frac{\partial E}{\partial r} + \frac{1}{r}\right)\frac{\partial u}{\partial r} + \left[\lambda\left(\frac{1}{E}\frac{\partial E}{\partial r}\right) - (\lambda + 2G)\frac{1}{r}\right]\frac{u}{r} + G\left(\frac{1}{E}\frac{\partial E}{\partial z}\right)\frac{\partial u}{\partial z} + G\left(\frac{\partial^2 u}{\partial z^2}\right) + G\left(\frac{1}{E}\frac{\partial E}{\partial z}\right)\frac{\partial w}{\partial r} + \lambda\left(\frac{1}{E}\frac{\partial E}{\partial r}\right)\frac{\partial w}{\partial z} + (\lambda + G)\frac{\partial^2 w}{\partial r \partial z} + F_r - \frac{1}{1 - 2\nu}\frac{\partial}{\partial r}(E\alpha T) = 0 \qquad (1)$$

$$(\lambda + 2G)\frac{\partial^2 w}{\partial z^2} + (\lambda + 2G)\left(\frac{1}{E}\frac{\partial E}{\partial z}\right)\frac{\partial w}{\partial z} + G\left(\frac{1}{E}\frac{\partial E}{\partial r} + \frac{1}{r}\right)\frac{\partial w}{\partial r} + (G)\frac{\partial^2 w}{\partial r^2} + \left[G\left(\frac{1}{E}\frac{\partial E}{\partial r}\right) + (\lambda + G)\frac{1}{r}\right]\frac{\partial u}{\partial z} + \left(\frac{1}{E}\frac{\partial E}{\partial z}\right)\frac{u}{r} + \lambda\left(\frac{1}{E}\frac{\partial E}{\partial z}\right)\frac{\partial u}{\partial r} + (\lambda + G)\frac{\partial^2 u}{\partial r \partial z} + F_z - \frac{1}{1 - 2\nu}\frac{\partial}{\partial z}(E\alpha T) = 0 \qquad (2)$$

The corresponding equations for a thin plate of variable thickness (Fig. 2) can be developed independently or can be deduced from Eqs. (1) and (2) by letting r go to infinity and determining the modifications of the elastic constants that result from approximating the plate as a plane stress problem. Thus, E must be replaced by Et in all terms, and the other constants must be modified to reflect the condition that the stress σ_t be equal to zero. The final equations are

$$\frac{Et}{1 - \nu^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{1 - \nu^2}\frac{\partial(Et)}{\partial r}\frac{\partial u}{\partial r} + \frac{1}{2(1 + \nu)}\frac{\partial(Et)}{\partial z}\frac{\partial u}{\partial z} + \frac{Et}{2(1 + \nu)}\frac{\partial^2 u}{\partial z^2} + \frac{1}{2(1 + \nu)}\frac{\partial(Et)}{\partial z}\frac{\partial w}{\partial r} + \frac{\nu}{1 - \nu^2}\frac{\partial(Et)}{\partial r}\frac{\partial w}{\partial z} + \frac{Et}{2(1 - \nu)}\frac{\partial^2 w}{\partial r \partial z} + t F_r - \frac{1}{1 - \nu}\frac{\partial}{\partial r}(Et\alpha T) = 0 \qquad (3)$$

$$\frac{Et}{1 - \nu^2}\frac{\partial^2 w}{\partial z^2} + \frac{1}{1 - \nu^2}\frac{\partial(Et)}{\partial z}\frac{\partial w}{\partial z} + \frac{1}{2(1 + \nu)}\frac{\partial(Et)}{\partial r}\frac{\partial w}{\partial r} + \frac{Et}{2(1 + \nu)}\frac{\partial^2 w}{\partial r^2} + \frac{1}{2(1 + \nu)}\frac{\partial(Et)}{\partial r}\frac{\partial u}{\partial z} + \frac{\nu}{1 - \nu^2}\frac{\partial(Et)}{\partial z}\frac{\partial u}{\partial r} + \frac{Et}{2(1 - \nu)}\frac{\partial^2 u}{\partial r \partial z} + tF_z - \frac{1}{1 - \nu}\frac{\partial}{\partial z}(Et\alpha T) = 0 \qquad (4)$$

There are several advantages in the choice of deformations as unknowns in the formulation of these differential equations. As pointed out by Allen[14] in discussing extension of a flat plate, the case of mixed boundary conditions can be solved more easily in terms of displacements than in terms of a stress function ϕ, and this is no less true in three-dimensional axisymmetric problems. The difficulties inherent in a fourth-order stress function formulation of isothermal axisymmetric problems have been discussed by Conte, Miller,

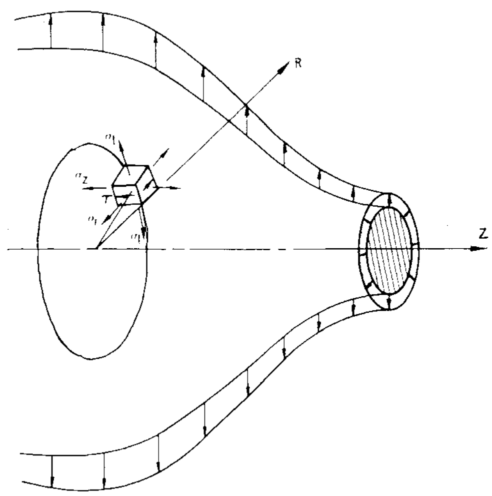

Fig. 1 Stressed element in axially symmetric solid.

and Sensenig.[2] For the more general problems considered here, even a stress function formulation patterned after the method of Southwell[14,15] could well prove impossible of solution.

Another advantage in the choice of displacements lies in the resulting similarity of mathematical form between the sets of equations (1–4). Thus, defining new constants and load terms, these equations can be replaced by a single set of two elliptic partial differential equations sufficiently general as to include the three-dimensional axisymmetric and two-dimensional extension problems as special cases. Considering only the body force due to a possible rotation about the z axis, these equations are

$$C_1\frac{\partial^2 u}{\partial r^2} + C_1\left[\frac{1}{E^*}\frac{\partial E^*}{\partial r} + R_1(r)\right]\frac{\partial u}{\partial r} + R_1(r)\left[C_2\frac{1}{E^*}\frac{\partial E^*}{\partial r} - C_1R_1(r)\right]u + C_3\left(\frac{1}{E^*}\frac{\partial E^*}{\partial z}\right)\frac{\partial u}{\partial z} + C_3\frac{\partial^2 u}{\partial z^2} + C_3\left(\frac{1}{E^*}\frac{\partial E^*}{\partial z}\right)\frac{\partial w}{\partial r} + C_2\left(\frac{1}{E^*}\frac{\partial E^*}{\partial r}\right)\frac{\partial w}{\partial z} + (C_2 + C_3)\frac{\partial^2 w}{\partial r \partial z} + \frac{1}{E}\rho\omega^2 r - C_2\left(\frac{1 + \nu}{\nu}\right)\frac{1}{E^*}\frac{\partial}{\partial r}(E^*\alpha T) = 0 \qquad (5)$$

$$C_1\frac{\partial^2 w}{\partial z^2} + C_1\left(\frac{1}{E^*}\frac{\partial E^*}{\partial z}\right)\frac{\partial w}{\partial z} + C_3\left[\frac{1}{E^*}\frac{\partial E^*}{\partial r} + R_1(r)\right]\frac{\partial w}{\partial r} + C_3\frac{\partial^2 w}{\partial r^2} + \left[C_3\frac{1}{E^*}\frac{\partial E^*}{\partial r} + (C_2 + C_3)R_1(r)\right]\frac{\partial u}{\partial z} + C_2 R_1(r)\left(\frac{1}{E^*}\frac{\partial E^*}{\partial z}\right)u + C_2\left(\frac{1}{E^*}\frac{\partial E^*}{\partial z}\right)\frac{\partial u}{\partial r} + (C_2 + C_3)\frac{\partial^2 u}{\partial r \partial z} - C_2\left(\frac{1 + \nu}{\nu}\right)\frac{1}{E^*}\frac{\partial}{\partial z}(E^*\alpha T) = 0 \qquad (6)$$

For axisymmetric solids

$$C_1 = \frac{1 - \nu}{(1 + \nu)(1 - 2\nu)} \qquad C_2 = \frac{\nu}{(1 + \nu)(1 - 2\nu)} \qquad C_3 = 1/[2(1 + \nu)] \qquad R_1(r) = 1/r \qquad E^* = 2(1 + \nu)G = E \qquad (7a)$$

For variable thickness plates (extensional problems),

$$C_1 = \frac{1}{1 - \nu^2} \qquad C_2 = \frac{\nu}{1 - \nu^2} \qquad C_3 = \frac{1}{2(1 + \nu)} \qquad R_1(r) = 0 \qquad E^* = 2(1 + \nu)Gt = Et \qquad (7b)$$

(In both cases $C_1 = C_2 + 2C_3$.)

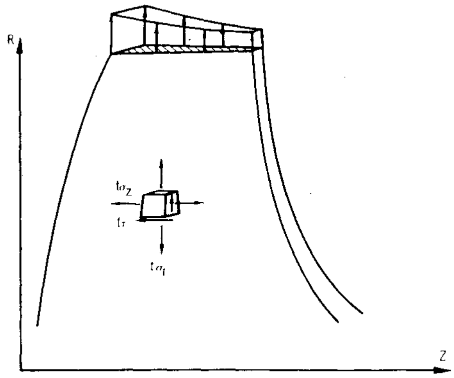

Fig. 2 Stressed element in variable thickness plate.

Equations (5) and (6) henceforth will be referred to as the governing differential equations of the two variable thermal and mechanical deformation problems. In the development of these equations the temperature rise T, the material properties E and α, and (for plate problems) the thickness t have been taken as variable. The material properties ρ and ν have been considered constant. The stresses σ_r, σ_t (axisymmetric solids only), σ_z, and τ, to be determined from the solution of Eqs. (5) and (6), are given by the expressions

$$\left.\begin{aligned}
\sigma_r &= C_1(E)(\partial u/\partial r) + C_2(E)[R_1(r)u + (\partial w/\partial z)] - C_2[(1+\nu)/\nu](E\alpha T)\\
\sigma_t &= C_1(E)R_1(r)u + C_2(E)[(\partial u/\partial r) + (\partial w/\partial z)] - C_2[(1+\nu)/\nu](E\alpha T)\\
\sigma_z &= C_1(E)(\partial w/\partial z) + C_2(E)[R_1(r)u + (\partial u/\partial r)] - C_2[(1+\nu)/\nu](E\alpha T)\\
\tau &= C_3(E)[(\partial u/\partial z) + (\partial w/\partial r)]
\end{aligned}\right\} \quad (8)$$

3. Finite-Difference Equations

The replacement of Eqs. (5) and (6) by finite-difference equations involves the determination of the difference approximations to all first- and second-order derivatives of u and w at each point of the network. At a regular point (Fig. 3), the well-known central difference approximations lead to the equations

$$\begin{aligned}
u_{i,j} = {} & C_1(1+K_1)(1/X)u_{i+1,j} + C_1(1-K_1)(1/X)u_{i-1,j} + C_3(1+K_2)(1/X)u_{i,j+1} + C_3(1-K_2)(1/X)u_{i,j-1} + {}\\
& C_3(K_2)(1/X)(w_{i+1,j} - w_{i-1,j}) + C_2(K_3)(1/X) \times (w_{i,j+1} - w_{i,j-1}) + [(C_2+C_3)/4](1/X)(w_{i+1,j+1} - {}\\
& w_{i+1,j-1} - w_{i-1,j+1} + w_{i-1,j-1}) + u_{i,j}^{\circ}
\end{aligned} \quad (9)$$

$$\begin{aligned}
w_{i,j} = {} & C_3(1+K_1)(1/Y)w_{i+1,j} + C_3(1-K_1)(1/Y)w_{i-1,j} + C_1(1+K_2)(1/Y)w_{i,j+1} + C_1(1-K_2)(1/Y)w_{i,j-1} + {}\\
& C_2(K_2)(1/Y)(u_{i+1,j} - u_{i-1,j}) + (C_2K_4 + C_3K_1) \times (1/Y)(u_{i,j+1} - u_{i,j-1}) + [(C_2+C_3)/4](1/Y) \times {}\\
& (u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}) + 4C_2(K_2K_4)(1/Y)u_{i,j} + w_{i,j}^{\circ}
\end{aligned} \quad (10)$$

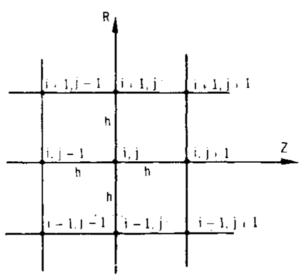

Fig. 3 Regular interior point.

where

$$\left.\begin{aligned}
K_1 &= \left(\frac{1}{E^*}\frac{\partial E^*}{\partial r} + R_1\right)\frac{h}{2} = K_3 + K_4 \qquad K_4 = R_1\frac{h}{2}\\
K_2 &= \left(\frac{1}{E^*}\frac{\partial E^*}{\partial z}\right)\frac{h}{2} \qquad K_3 = \left(\frac{1}{E^*}\frac{\partial E^*}{\partial r}\right)\frac{h}{2}\\
u_{i,j}^{\circ} &= \left(\frac{1}{E}\rho\omega^2 r\right)\frac{h^2}{X} - C_2\left(\frac{1+\nu}{\nu}\right)\frac{1}{E^*}\frac{\partial}{\partial r}(E^*\alpha T)\frac{h^2}{X}\\
w_{i,j}^{\circ} &= -C_2\left(\frac{1+\nu}{\nu}\right)\frac{1}{E^*}\frac{\partial}{\partial z}(E^*\alpha T)\frac{h^2}{X}\\
X &= 2(C_1 + C_2) - 4(C_2K_3 - C_1K_4)K_4\\
Y &= 2(C_1 + C_3)
\end{aligned}\right\} \quad (11)$$

K_1, K_2, K_3, K_4, X, and Y are dimensionless variables determined from problem geometry, thermal gradients, and material property vs temperature curves. $u_{i,j}^{\circ}$ and $w_{i,j}^{\circ}$, having the dimension of length, are load terms (starting values) that depend upon mechanical (centrifugal) loading as well as upon these same factors of geometry, temperature, and material.

For an irregular point, the simplest approach is to choose a five-point neighborhood and approximate the derivatives as the corresponding rates of change of a polynomial surface passing through the point and the neighborhood. This can be accomplished by a formal Taylor's series expansion of either of the deformations u or w about the point, dropping terms of third and higher degree. For a typical point near the boundary in the upper right quadrant of an irregular domain (Fig. 4), the resulting finite-difference equations are

$$\begin{aligned}
u_{i,j} = {} & \frac{2C_1}{\xi(1+\xi)}(1+K_1)\frac{1}{X}u_T + \frac{2C_1}{1+\xi} \times (1-\xi K_1)\frac{1}{X}u_{i-1,j} + \frac{2C_3}{\eta(1+\eta)}(1+K_2)\frac{1}{X}u_R + {}\\
& \frac{2C_3}{1+\eta}(1-\eta K_2)\frac{1}{X}u_{i,j-1} + \frac{2C_3}{\xi(1+\xi)}(K_2)\frac{1}{X} \times (w_T - \xi^2 w_{i-1,j}) + \frac{2C_2}{\eta(1+\eta)}(K_3)\frac{1}{X}(w_R - \eta^2 w_{i,j-1}) + {}\\
& (C_2 + C_3)\frac{1}{X}(w_{i,j} - w_{i-1,j} - w_{i,j-1} + w_{i-1,j-1}) - {}\\
& \left(2C_2K_3\frac{1-\eta}{\eta} + 2C_3K_2\frac{1-\xi}{\xi}\right)\frac{1}{X}w_{i,j} + u_{i,j}^{\circ}
\end{aligned} \quad (12)$$

$$\begin{aligned}
w_{i,j} = {} & \frac{2C_3}{\xi(1+\xi)}(1+K_1)\frac{1}{Y}w_T + \frac{2C_3}{1+\xi}(1-\xi K_1) \times \frac{1}{Y}w_{i-1,j} + \frac{2C_1}{\eta(1+\eta)}(1+K_2)\frac{1}{Y}w_R + \frac{2C_1}{1+\eta} \times {}\\
& (1-\eta K_2)\frac{1}{Y}w_{i,j-1} + \frac{2C_2}{\xi(1+\xi)}(K_2)\frac{1}{Y}(u_T - \xi^2 u_{i-1,j}) + {}\\
& \frac{2}{\eta(1+\eta)}(C_2K_4 + C_3K_1)(1/Y)(u_R - \eta^2 u_{i,j-1}) + {}\\
& (C_2 + C_3)\frac{1}{Y}(u_{i,j} - u_{i-1,j} - u_{i,j-1} + u_{i-1,j-1}) + {}\\
& \left[4C_2K_2K_4 - 2C_2K_2\frac{1-\xi}{\xi} - 2(C_2K_4 + C_3K_1)\frac{1-\eta}{\eta}\right] \times \frac{1}{Y}u_{i,j} + w_{i,j}^{\circ}
\end{aligned} \quad (13)$$

where

$$\begin{aligned}
X &= 2C_1\left(\frac{1}{\xi} + K_1\frac{1-\xi}{\xi}\right) + 2C_3\left(\frac{1}{\eta} + K_2\frac{1-\eta}{\eta}\right) - 4(C_2K_3 - C_1K_4)K_4\\
Y &= 2C_1\left(\frac{1}{\eta} + K_2\frac{1-\eta}{\eta}\right) + 2C_3\left(\frac{1}{\xi} + K_1\frac{1-\xi}{\xi}\right)
\end{aligned} \quad (14)$$

and the other terms are as defined in Eqs. (11).

The absence from Eqs. (12) and (13) of fictitious points outside the boundary is significant to the development of a computer program capable of forming and solving the difference equations for a general boundary configuration. Similar equations can be written for points adjacent to the boundary in the other quadrants of the domain, thus completing a discretization free from dependence upon fictitious points. These equations for the displacements have errors of order h^3.

4. Boundary Conditions

Boundary conditions in which tractions and/or displacements are specified can be classified into four groups. Referring to Fig. 5, these classifications are 1) displacements u and w prescribed, 2) tractions R and Z prescribed, 3) normal displacement Δ_N and tangential shearing stress τ_T prescribed, and 4) normal stress σ_N and tangential displacement Δ_T prescribed.

The corresponding equations written for boundary point 0 (Fig. 5) and expressed in terms of the displacements u and w, follow:

Class 1

$$u_0 = F_1 \qquad w_0 = F_2 \tag{15}$$

Class 2

$$\left.\begin{aligned} &C_1 l\left(\frac{\partial u}{\partial r}\right)_0 + C_2 l\left(R_1(r)u + \frac{\partial w}{\partial z}\right)_0 + \\ &\qquad C_3 m\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)_0 - C_2 l\,\frac{1+\nu}{\nu}(\alpha T)_0 = \frac{F_1}{E} \\ &C_1 m\left(\frac{\partial w}{\partial z}\right)_0 + C_2 m\left(R_1(r)u + \frac{\partial u}{\partial r}\right)_0 + \\ &\qquad C_3 l\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)_0 - C_2 m\,\frac{1+\nu}{\nu}(\alpha T)_0 = \frac{F_2}{E} \end{aligned}\right\} \tag{16}$$

Class 3

$$\left.\begin{aligned} &lu_0 + mw_0 = F_1 \\ &(C_1 - C_2)lm(\partial w/\partial z)_0 - (C_1 - C_2)lm(\partial u/\partial r)_0 + \\ &C_3(l^2 - m^2)(\partial u/\partial z)_0 + C_3(l^2 - m^2)(\partial w/\partial r)_0 = F_2/E \end{aligned}\right\} \tag{17}$$

Class 4

$$\left.\begin{aligned} &lw_0 - m\,u_0 = F_1 \\ &(C_1 l^2 + C_2 m^2)(\partial u/\partial r)_0 + (C_1 m^2 + C_2 l^2)(\partial w/\partial z)_0 + \\ &C_2 R_1(r)u_0 + 2C_3 lm(\partial u/\partial z + \partial w/\partial r)_0 - \\ &\qquad C_2(1 + \nu/\nu)(\alpha T)_0 = F_2/E \end{aligned}\right\} \tag{18}$$

where l and m are the direction numbers of the outward normal with respect to the R and Z axes, and F_1 and F_2 are the prescribed boundary values.

The first class of boundary conditions (that of prescribed displacements) requires no further discussion. The boundary conditions of types 2, 3, and 4, however, involve first-order partial differential equations in the displacements and consequently require the development of corresponding difference approximations. In the simpler problem of normal gradient boundary conditions, a number of approximation procedures have been suggested, many of which involve the use of fictitious points. One of the better known approximations is attributed to Fox,[16] and an excellent method has been contributed recently by Allen and Robins.[17] Other approximative techniques for the normal gradient problem are described by Crandall,[18] Forsythe and Wasow,[19] and Allen.[14] These various procedures are not completely suitable for the more general equations (16–18), however, and if fictitious points are to be avoided altogether, another technique must be developed. The theory and method of approximation advanced herein was given by the author in 1963[20] and independently proposed by Greenspan[21] in 1964

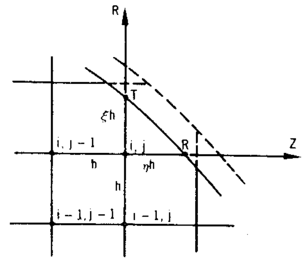

Fig. 4 Irregular point near boundary.

(although the latter restricted consideration to normal gradient conditions in one dependent variable). It is a direct method of quadratic approximation suitable for any set of first-order partial differential boundary equations in two dependent variables. The application of this method to the boundary equations (16–18) follows.

At a typical boundary point 0 (Fig. 5), the displacement u (or w) is expanded in a Taylor's series:

$$u = u_0 + x\left(\frac{\partial u}{\partial r}\right)_0 + y\left(\frac{\partial u}{\partial z}\right)_0 + \frac{x^2}{2}\left(\frac{\partial^2 u}{\partial r^2}\right)_0 + xy\left(\frac{\partial^2 u}{\partial r \partial z}\right)_0 + \frac{y^2}{2}\left(\frac{\partial^2 u}{\partial z^2}\right)_0 + \ldots \tag{19}$$

where x and y are coordinates relative to the boundary point in the R and Z directions, respectively. Evaluating this equation at each point of an irregular neighborhood of points $1, \ldots, 5$ (Fig. 5), and denoting the elements of the resulting coordinates matrix by α_{kl}, the following matrix equations can be written:

$$[\alpha_{kl}]\{\partial u\} = \{u_k - u_0\} \qquad [\alpha_{kl}]\{\partial w\} = \{w_k - w_0\} \tag{20}$$

where $\{\partial u\}$ and $\{\partial w\}$ represent the column vectors of the partial derivatives of u and w, respectively. From the solution of Eqs. (20), the expressions for the first-order derivatives in terms of the inverse matrix elements β_{lk} are

$$\left.\begin{aligned} \left(\frac{\partial u}{\partial r}\right)_0 &= \sum_1^5 \beta_{1k}u_k - u_0\sum_1^5 \beta_{1k} \\ \left(\frac{\partial u}{\partial z}\right)_0 &= \sum_1^5 \beta_{2k}\,u_k - u_0\sum_1^5 \beta_{2k} \\ \left(\frac{\partial w}{\partial r}\right)_0 &= \sum_1^5 \beta_{1k}\,w_k - w_0\sum_1^5 \beta_{1k} \\ \left(\frac{\partial w}{\partial z}\right)_0 &= \sum_1^5 \beta_{2k}\,w_k - w_0\sum_1^5 \beta_{2k} \end{aligned}\right\} \tag{21}$$

By expressing the coordinates of the general point k as $\xi_k h$, $\eta_k h$, and writing Eq. (19) in the form

$$u_k = u_0 + \xi_k h\left(\frac{\partial u}{\partial r}\right)_0 + \eta_k h\left(\frac{\partial u}{\partial z}\right)_0 + \frac{1}{2}\,\xi_k{}^2h^2\left(\frac{\partial^2 u}{\partial r^2}\right)_0 + \xi_k\eta_k h^2\left(\frac{\partial^2 u}{\partial r\partial z}\right)_0 + \frac{1}{2}\,\eta_k{}^2h^2\left(\frac{\partial^2 u}{\partial z^2}\right)_0 + 0(h^3) \tag{22}$$

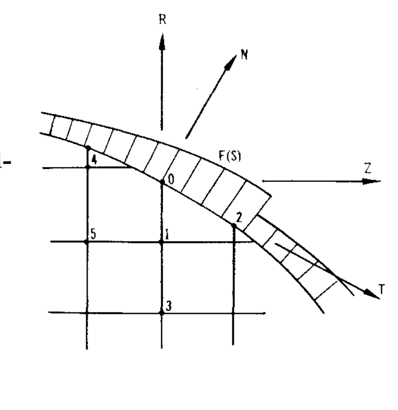

Fig. 5 General boundary point.

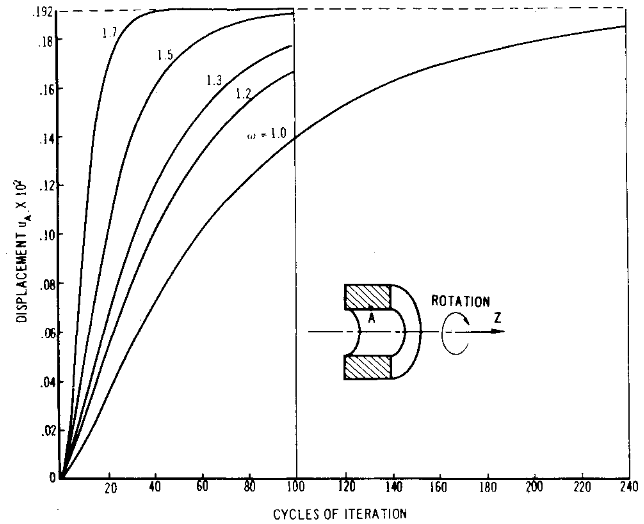

Fig. 6 Deformation vs iteration curves for various overrelaxation factors.

it can be shown that the errors in the difference approximations of Eqs. (21) are of order h^2. Thus, the errors in the displacements are of order h^3, consistent with Eqs. (12) and (13) for interior points.

The final finite-difference approximations to the boundary conditions of types 2, 3, and 4 are obtained by substituting Eqs. (21) into the differential equations (16–18), solving simultaneously for u_0 and w_0 in each case. The resulting equations, expressed in general form, are

$$\begin{aligned} u_0 &= \sum_1^5 A_{1k}\, u_k + \sum_1^5 A_{2k}\, w_k + u_0{}^* \\ w_0 &= \sum_1^5 B_{1k}\, u_k + \sum_1^5 B_{2k}\, w_k + w_0{}^* \end{aligned} \tag{23}$$

where

$$\left.\begin{aligned} A_{jk} &= (1/X_0)(G_2\, D_{jk} - D_2 G_{jk}) \\ B_{jk} &= (1/X_0)(D_1 G_{jk} - G_1\, D_{jk}) \\ u_0{}^* &= (1/X_0)(G_2 D_0 - D_2 G_0) \\ w_0{}^* &= (1/X_0)(D_1 G_0 - G_1 D_0) \end{aligned}\right\} \tag{24}$$

$$X_0 = D_1 G_2 - D_2 G_1 \tag{25}$$

For boundary condition class 2 (prescribed tractions), the D and G quantities in Eqs. (24) are given by the expressions

$$\left.\begin{aligned} D_1 &= C_1 l \sum_1^5 \beta_{1k} - C_2 l R_1 + C_3 m \sum_1^5 \beta_{2k} \\ D_2 &= C_2 l \sum_1^5 \beta_{2k} + C_3 m \sum_1^5 \beta_{1k} \\ G_1 &= C_2 m \sum_1^5 \beta_{1k} - C_2 m R_1 + C_3 l \sum_1^5 \beta_{2k} \\ G_2 &= C_1 m \sum_1^5 \beta_{1k} + C_3 l \sum_1^5 \beta_{1k} \\ D_{1k} &= C_1 l \beta_{1k} + C_3 m \beta_{2k} \quad G_{1k} = C_2 m \beta_{1k} + C_3 l \beta_{2k} \\ D_{2k} &= C_2 l \beta_{2k} + C_3 m \beta_{1k} \quad G_{2k} = C_1 m \beta_{2k} + C_3 l \beta_{1k} \\ D_0 &= -C_2 l[(1+\nu)/\nu](\alpha T)_0 - (F_1/E) \\ G_0 &= -C_2 m[(1+\nu)/\nu](\alpha T)_0 - (F_2/E) \end{aligned}\right\} \tag{26}$$

For boundary condition class 3 the corresponding equations are

$$\left.\begin{aligned} D_1 &= l \qquad D_2 = m \\ G_1 &= (C_1 - C_2) lm \sum_1^5 \beta_{1k} - C_3(l^2 - m^2) \sum_1^5 \beta_{2k} \\ G_2 &= -(C_1 - C_2) lm \sum_1^5 \beta_{2k} - C_3(l^2 - m^2) \sum_1^5 \beta_{1k} \\ D_{1k} &= 0 \qquad D_{2k} = 0 \\ G_{1k} &= (C_1 - C_2) lm \beta_{1k} - C_3(l^2 - m^2)\beta_{2k} \\ G_{2k} &= -(C_1 - C_2)\, lm \beta_{2k} - C_3(l^2 - m^2)\beta_{1k} \\ D_0 &= F_1 \qquad G_0 = F_2/E \end{aligned}\right\} \tag{27}$$

with similar equations for class 4.

5. Matrix Solution

The finite-difference equations for the displacements u and w at any point p, either in the interior of the problem domain or on the boundary S, can be expressed in general form as

$$\begin{aligned} u_p &= \sum_q a_{pq} u_q + \sum_q b_{pq} w_q + u_p{}^\circ \\ w_p &= \sum_q c_{pq} w_q + \sum_q d_{pq} u_q + w_p{}^\circ \end{aligned} \tag{28}$$

where the coefficients $a_{pq}, \ldots, d_{pq}$ and the load terms $u_p{}^\circ$, $w_p{}^\circ$ are determined from the appropriate equations (9, 10, 12, 13, 23, and 24, etc.). In matrix form, Eqs. (28) become

$$[A]\{\delta\} = \{\delta^\circ\} \tag{29}$$

with the order N of the square coefficients matrix $[A]$ equal to the total number of unknown displacements u and w. The column vector of these displacements is $\{\delta\}$, and $\{\delta^\circ\}$ is the loads vector of starting values u°, w°. From Eqs. (28) it is evident that the diagonal elements A_{pp} of the $[A]$ matrix are unity, whereas each off-diagonal element A_{pq} is equal in magnitude but opposite in sign to the corresponding coefficient a_{pq}, b_{pq}, etc.

Matrix $[A]$ is a sparse matrix of large order N with a maximum of fourteen nonzero elements in any one row. Thus, although the matrix $[A]$ does not have the desirable properties of symmetry and nonnegativeness, it nevertheless possesses characteristics that suggest an iterative method of solution. Of those cyclic iterative techniques suitable for computer computation, the one chosen herein (and one of the more widely used) is the method of successive overrelaxation, the use of systematic overrelaxation with the method of successive displacements. The value of this method was theoretically established by Young[22] for a class of matrices having what is standardly termed property (A), but it has been demonstrated (as here) to extend to considerably more general matrices.

Denoting the overrelaxation factor by ω, the general equation of successive overrelaxation is given as

$$\omega \sum_{q<p} A_{pq}\delta_q{}^{n+1} + \delta_p{}^{n+1} + \omega \sum_{q>p} A_{pq}\delta_q{}^n = (1-\omega)\delta_p{}^n + \omega\delta_p{}^\circ \tag{30}$$

where $\delta_p{}^{n+1}$ is the $n + 1$st corrected value of δ_p. It is doubtful that successive overrelaxation can be proved convergent for a general matrix $[A]$ formed from a direct discretization of the field equations and a quadratic approximation of the boundary conditions. The author has found the method to converge, however, for a wide range of both thermoelastic and isothermal problems in plates and axisymmetric solids, considering high-strength steels having a Poisson's ratio approximately equal to 0.3 and using values of ω from 1.5 to 1.7. The divergence (for circular cylinders) reported by

Conte, Miller, and Sensenig[2] possibly can be attributed to their consideration of a solid-propellant-like material having a Poisson's ratio near 0.5. (They do not mention this point in their paper, however.)

6. Computer Program and Numerical Applications

A FORTRAN language computer program has been developed[20] that is capable of performing the finite-difference discretization and solution of the general deformation boundary value problem considered herein. The program input includes a description of the boundary configuration, the problem type, material properties, rotational speed and/or temperature distribution, thickness variation (for extensional problems), and boundary conditions and prescribed boundary functions. The output includes network structure and boundary geometry (phase 1), matrix coefficients and starting values (phase 2), and final deformations and stresses (phases 3 and 4). The program is designed for an IBM 7090 or 7094 data processing system with 32K core storage. The maximum size network permitted consists of approximately 1000 modal points over a rectangular domain (requiring an order of 26,000 data locations during iteration), with fewer effective points in cases of irregular configurations.

For a typical mixed boundary value problem, the first two phases of the computation require a total of 1–$1\frac{1}{2}$ min. The last phase, calculation of the stresses, requires only a few seconds. The major portion of the time required for a problem run is taken up by the iteration phase; thus a well-chosen overrelaxation factor can result in a considerable economy. This is strikingly illustrated in Fig. 6 wherein curves of deformation vs cycles of iteration are depicted for various overrelaxation factors from 1.0 to 1.7. The deformation values given correspond to the radial displacement u at point A of a thick disk rotating about the Z axis. Note that, for an overrelaxation factor $\omega = 1.7$, the iteration has converged to three significant figures in less than 50 cycles, whereas for the method of successive displacements without overrelaxation ($\omega = 1$), the value of u_A is still more than 4% in error after 240 cycles. (A very coarse mesh was used for this problem, corresponding to approximately 65 points over the cross section.)

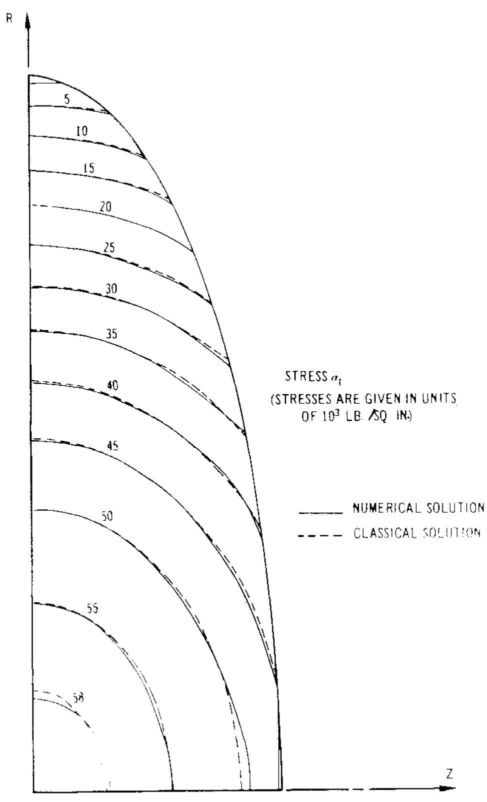

Fig. 7 Comparison of numerical and classical solutions for radial stress distribution in rotating ellipsoid.

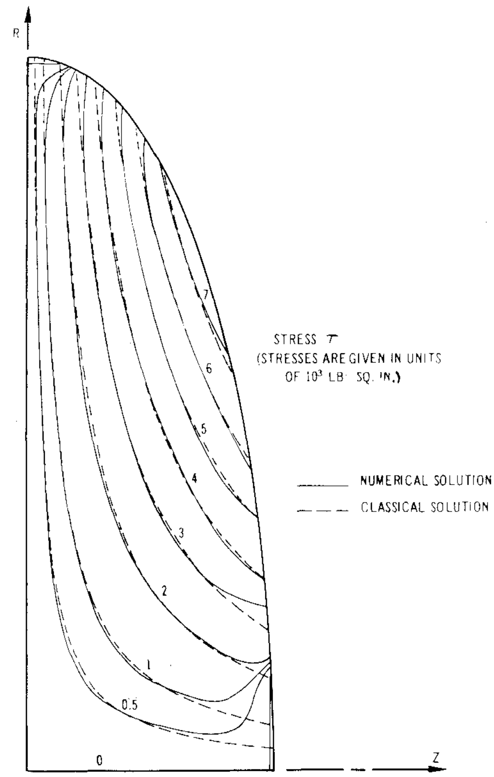

Fig. 8 Comparison of numerical and classical solutions for shearing stress distribution in rotating ellipsoid.

For overrelaxation factors between 1.0 and 1.7, the rate of convergence ranged between these two extremes as shown (Fig. 6). When a solution was attempted with a factor of 1.8, however, the iteration diverged. Similarly, for the various other problems that have been calculated by the program, the iteration diverged for overrelaxation factors that were sufficiently large but converged at increasing rates with increasing ω for smaller values (a result also reported by Thoms[1] for the problem of a variable diameter cylinder under internal pressure). Thus, it has been observed that the optimum overrelaxation factor must be quite close to the largest value for which convergence can be obtained, the optimum and maximum values having been found to lie between 1.7 and 1.8 for isothermal problems with simple rectangular boundaries, between 1.6 and 1.7 for isothermal problems with curved or irregular boundaries, and between 1.5 and 1.6 for thermal stress problems with irregular boundaries.

A comparison of numerical results from the finite-difference computer solution of a rotating ellipsoid with the classical solution obtained by Chree[23] is shown in Figs. 7 and 8. The ellipsoid is 6 in. in diameter and 2 in. in breadth, rotating at a speed of 50,000 rpm; 300 cycles of iteration were performed, using an overrelaxation factor of 1.6. A total running time of 12 min. was required for the four phases of the computation. (A network of medium fineness consisting of 26 horizontal grid lines and 8 vertical grid lines over one quarter of the ellipse was chosen, resulting in 240 network points and 446 displacement unknowns.) After 290 cycles, the u displacements were essentially converged to three significant figures, and at the end of 300 cycles the maximum change per cycle in the w displacements had decreased to 0.01725%. The deviation in the shearing stress values from the classical solution near the axes of the ellipsoid, as seen in Fig. 8, was

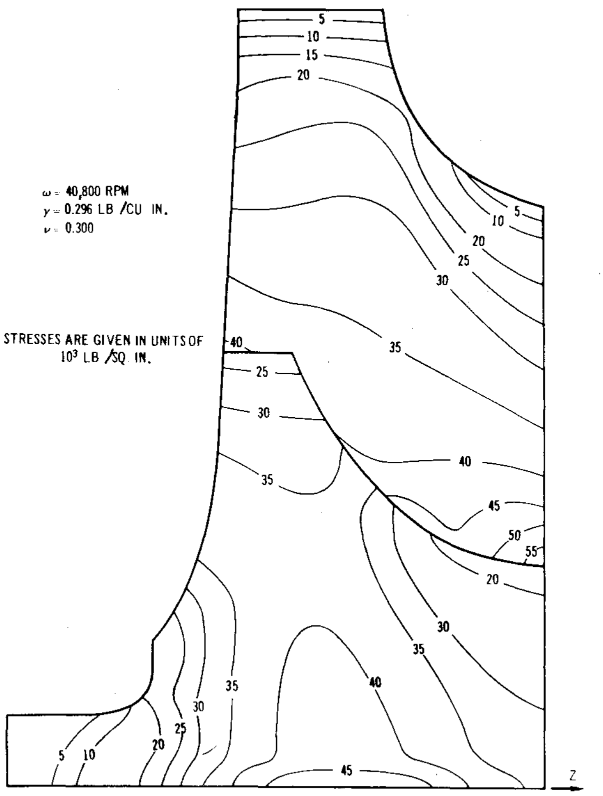

Fig. 9 Radial stress distribution in turbine wheel blade and disk.

caused by "slicing off" the body in these regions. Thus, the discrete model is slightly flat at the top and the side, with the corresponding free-surface boundary condition of zero tangential shearing stress imposed along these boundary segments. Agreement with the classical solution for the circumferential stress σ_t and the small axial stress σ_z is comparable to that for σ_r and τ, with σ_z deviating slightly more because of the small increments in w still being experienced after 300 cycles.

The calculated radial stress distribution in a typical small radial-flow turbine wheel is given in Fig. 9. The wheel is approximately 9 in. in diameter with a maximum disk radius of 2.6 in. The upper or blade portion of the wheel cross section is a variable thickness plate, relatively thin along its entire right edge. The various stresses and deformations in the wheel were obtained by separate computations of the blade and disk, performing 400 cycles of iteration (with ω = 1.6) in each problem and using a total of 45–50 min of machine time for the two solutions taken together. A network consisting of 27 horizontal and 15 vertical grid lines was chosen for the blade, with a network of 21 by 25 lines chosen for the disk (giving a combined total of 1516 unknown displacements). In the blade analysis a segment of the disk was considered to be acting with the blade, and the resulting body was isolated and taken as fixed along its curved lower boundary. The force intensities along the actual blade-disk junction, obtained from the solution of this problem as a variable thickness plate, were applied as boundary values to the disk through the equations

$$
\begin{aligned}
2\pi\, r_s\, F_1^{\text{disk}} + n\, t_s^{\text{blade}}\, R^{\text{blade}} &= 0 \\
2\pi\, r_s\, F^{\text{disk}} + n\, t_s^{\text{blade}}\, Z^{\text{blade}} &= 0
\end{aligned}
\qquad (31)
$$

where the subscript s denotes the blade-disk common boundary, n is the number of blades, and R and Z are the boundary force intensities for the blade. The disk problem was then solved as an axisymmetric solid. Thus, the force equations along the blade-disk boundary are satisfied by the combined stress distribution in the turbine wheel, but the displacement compatiblity equations are not exactly satisfied. Consequently the radial stresses given in Fig. 9 should properly be considered as preliminary values only. However, the lack of perfect compatibility of deformations along the blade-disk boundary may not be particularly significant in a problem such as this one. The stress resultants over the boundary satisfy static equilibrium of the blade, and small variations in the distribution of these values should have only localized effect, since all radial stresses are tensile, and there are no large stress gradients near the boundary in either the blade or the disk. (The three dimensional, nonaxisymmetric true nature of the problem near the blade-disk junction is not considered here, being beyond the scope of this analysis.)

In the case of thermal loading of the turbine wheel of Fig. 9, corresponding to a temperature gradient of approximately 800°F along the axis and a total variation of 1200°F from the end of the shaft to the blade tip, separate solutions of the blade and disk are less satisfactory.[24] Deformation incompatibilities of significant magnitude are introduced, leading to unrealistically large bending stresses in the disk. To enable the solution of such combined blade-disk problems a new computer program has been developed, permitting the simultaneous solution of a variable thickness plate and an (ideally) axisymmetric solid. The application of this program to thermal and centrifugal problems in turbine wheels is now in progress.

7. Conclusions

A formulation and finite-difference discretization of the general elliptic partial differential equations governing thermal and mechanical deformation problems in plates and axisymmetric solids has been presented. Mixed boundary conditions on irregular configurations have been considered, and a new method of quadratic approximation applicable to any set of first-order partial differential equations in two dependent variables has been advanced. From a comparison with the classical solution of a rotating ellipsoid, the methods of discretization have been found highly satisfactory.

The successive overrelaxation method has been used within the computer program described as the means of solving the general matrix equation. Although it is doubtful that convergence of the method can be theoretically established in the general case, it has been demonstrated herein to be an appropriate technique of solution in several important problems from the turbomachinery field.

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